Nuclear & Particle Physics of Compact Stars

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How Neutron Stars are Formed

Equations of Stellar Structure-I

- In hydrostatic equilibrium, the structure of a spherically symmetric neutron star from the Tolman-Oppenheimer-Volkov (TOV) equations:

\[
\frac{dM(r)}{dr} = 4\pi r^2 \epsilon(r)
\]

\[
\frac{dP(r)}{dr} = -\frac{GM(r)\epsilon(r)}{c^2 r^2} \left[ 1 + \frac{P(r)}{\epsilon(r)} \right] \left[ 1 + \frac{4\pi r^3 P(r)}{M(r)c^2} \right] \left[ 1 - \frac{2GM(r)}{c^2 r} \right]
\]

- \(G\) := Gravitational constant
- \(P\) := Pressure
- \(\epsilon\) := Energy density
- \(M(r)\) := Enclosed gravitational mass
- \(R_s = 2GM/c^2\) := Schwarzschild radius
The gravitational and baryon masses of the star:

\[ M_G c^2 = \int_0^R dr \ 4\pi r^2 \ \epsilon(r) \]

\[ M_A c^2 = m_A \int_0^R dr \ 4\pi r^2 \ \frac{n(r)}{1 - \frac{2GM(r)}{c^2r}}^{1/2} \]

- \( m_A \) := Baryonic mass
- \( n(r) \) := Baryon number density

The binding energy of the star \( B.E. = (M_A - M_G)c^2 \).

To determine star structure:

- Specify equation of state, \( P = P(\epsilon) \)
- Choose a central pressure \( P_c = P(\epsilon_c) \) at \( r = 0 \)
- Integrate the 2 DE’s out to surface \( r = R \), where \( P(r = R) = 0 \).
Nucleonic Equation of State

- Energy ($E$) & Pressure ($P$) vs. scaled density ($u = n/n_0$).
- Nuclear matter equilibrium density $n_0 = 0.16 \text{ fm}^{-3}$.
- Proton fraction $x = n_p/(n_p + n_n)$.
- Nuclear matter: $x = 1/2$.
- Neutron matter: $x = 0$.
- Stellar matter in $\beta-$equilibrium: $x = \tilde{x}$. 
Results of Star Structure

- Stellar properties for soft & stiff (by comparison) EOS’s.
- Causal limit: \( P = \epsilon \).
- \( M_g \): Gravitational mass
- \( R \): Radius
- \( BE \): Binding energy
- \( n_b \): Central density
- \( I \): Moment of inertia
- \( \phi \): Surface red shift, \( e^{\phi/c^2} = (1 - 2GM/c^2R)^{-1/2} \).
Constraints on the EOS-I

- $R > R_s = 2GM/c^2 \Rightarrow \frac{M}{M_\odot} \geq \frac{R}{R_s \odot}$;
  $R_s \odot = 2GM_\odot/c^2 \approx 2.95 \text{ km}$.

- $P_c < \infty$
  $\Rightarrow R > (9/8)R_s$
  $\Rightarrow \frac{M}{M_\odot} \geq (8/9)R/R_{s\odot}$.

- Sound speed $c_s$:
  $c_s = (dP/d\epsilon)^{1/2} \leq c$
  $\Rightarrow R > 1.39R_s$
  $\Rightarrow \frac{M}{M_\odot} \geq \frac{R}{(1.39R_{s\odot})}$.

- If $P = \epsilon$ above
  $n_t \approx 2n_0$,
  $R > 1.52R_s \Rightarrow \frac{M}{M_\odot}R/(1.52R_{s\odot})$. 
Constraints on the EOS-II

- $M_{max} \geq M_{obs}$;
  In PSR 1913+16,
  $M_{obs} = 1.44 \, M_\odot$.

- In PSR 1957+20,
  $P_K = 1.56$ ms:
  $\Omega_K \simeq 7.7 \times 10^3$
  $$\left( \frac{M_{max}}{M_\odot} \right)^{1/2} \left( \frac{R_{max}}{10 \, \text{km}} \right)^{-3/2} \, \text{s}^{-1}$$

- Mom. of Inertia $I$:
  $I_{max} = 0.6 \times 10^{45} \, \text{g cm}^2$
  $$\left( \frac{M_{max}}{M_\odot} \right) \left( \frac{R_{max}}{10 \, \text{km}} \right)^2 f(M_{max}, R_{max})$$

- In SN 1987A
  $B.E. \simeq (1 - 2) \times 10^{53} \, \text{ergs.}$
Composition of Dense Stellar Matter

- **Crustal Surface**: 
electrons, nuclei, dripped neutrons, · · · set in a lattice
new phases with lasagna, sphagetti, · · · like structures

- **Liquid (Solid?) Core**: 

  \[ n, p, \Delta, \cdots \quad \text{leptons: } e^\pm, \mu^\pm, \nu_e', s, \nu_\mu' s \]
  \[ \Lambda, \Sigma, \Xi, \cdots \]
  \[ K^-, \pi^-, \cdots \text{ condensates} \]
  \( u, d, s, \cdots \text{ quarks} \)

- **Constraints**:
  1. \( n_b = n_n + n_p + n_\Lambda + \cdots \) : baryon \# conservation
  2. \( n_p + n_\Sigma^+ + \cdots = n_e + n_\mu \) : charge neutrality
  3. \( \mu_i = b_i \mu_n - q_i \mu_\ell \) : energy conservation

\[ \mu_\Lambda = \mu_\Sigma^0 = \mu_\Xi^0 = \mu_n \]
\[ \mu_\Sigma^- = \mu_\Xi^- = \mu_n + \mu_e \]
\[ \mu_p = \mu_\Sigma^+ = \mu_n - \mu_e \]
\[ \mu_K^- = \mu_e = \mu_\mu = \mu_n - \mu_p \]
\[ \mu_d = \mu_u + \mu_e = \mu_s = (\mu_n + \mu_e)/3 \]
\[ \mu_u = (\mu_n - 2\mu_e)/3 \]
Consider equal numbers neutrons \((N)\) and protons \((Z)\) in a large volume \(V\) at zero temperature \((T = 0)\).

Let \(n = (N + Z)/V = n_n + n_p\) denote the neutron plus proton number densities; \(n = 2k_F^3/(3\pi^2)\), where \(k_F\) is the Fermi momentum.

Given the energy density \(\epsilon(n)\) inclusive of the rest mass density \(mn\), denote the energy per particle by \(E/A = \epsilon/n\), where \(A = N + Z\).

**Pressure:** From thermodynamics, we have

\[
P = - \frac{\partial E}{\partial V} = - \frac{dE}{d(A/n)}
\]

\[
= n^2 \frac{d(\epsilon/n)}{dn} = n \frac{d\epsilon}{dn} - \epsilon = n\mu - \epsilon,
\]

where \(\mu = d\epsilon/dn\) is the chemical potential inclusive of the rest mass \(m\). At the equilibrium density \(n_0\), where \(P(n_0) = 0, \mu = \epsilon/n = E/A\).
Nuclear Matter-II

Incompressibility: The compressibility $\chi$ is usually defined by

$$\chi = -\frac{1}{V} \frac{\partial V}{\partial P} = \frac{1}{n} \left( \frac{dP}{dn} \right)^{-1}$$

However, in nuclear physics applications, the incompressibility factor

$$K(n) = 9 \frac{dP}{dn} = 9 n \frac{d^2\epsilon}{dn^2}, \quad \text{or}$$

$$= 9 \frac{d}{dn} \left[ n^2 \frac{d(E/A)}{dn} \right] = 9 \left[ n^2 \frac{d^2(E/A)}{dn^2} + 2n \frac{d(E/A)}{dn} \right]$$

is used. At the equilibrium density $n_0$, the compression modulus

$$K(n_0) = 9n_0^2 \left. \frac{d^2(E/A)}{dn^2} \right|_{n_0} = k_F^{9/2} \left. \frac{d^2(E/A)}{dk_F^2} \right|_{k_F^0}.$$  

Above, $k_F^0 = (3\pi^2 n_0/2)^{1/3}$ denotes the equilibrium Fermi momentum.
**Adiabatic sound speed:** The propagation of small scale density fluctuations occurs at the sound speed obtained from the relation

\[
\left( \frac{c_s}{c} \right)^2 = \frac{dP}{d\epsilon} = \frac{dP/dn}{d\epsilon/dn} = \frac{1}{\mu} \frac{dP}{dn} = \frac{d\ln \mu}{d\ln n}.
\]

Alternative relations for the sound speed squared are

\[
\left( \frac{c_s}{c} \right)^2 = \frac{K}{9\mu} = \Gamma \frac{P}{P + \epsilon},
\]

where \( \Gamma = d\ln P/d\ln \epsilon \) is the adiabatic index. It is desirable to require that the sound speed does not exceed that of light.
\[ \epsilon(n) = n \left[ m + \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} + \frac{A}{2} u + \frac{B}{\sigma + 1} u^\sigma \right], \]

\( m := \) nucleon mass,

\( u = n/n_0 := \) density compression ratio,

Second term := mean kinetic energy of isospin symmetric matter,

Last two terms := parametrize contributions from density dependent potential interactions. Such terms arise, for example, when local contact interactions are used to model the nuclear forces.

The coefficients \( A, B \) and \( \sigma \) can be determined by using the empirically determined properties of nuclear matter at the equilibrium density.
Schematic Nuclear Matter EOS-II

Using the relations for the state variables, and recalling that the total baryon density \( n = 2k_F^3/3\pi^2 \), we get

\[
\frac{E}{A} - m = \frac{\epsilon}{n} - m = \langle E_F^0 \rangle u^{2/3} + \frac{A}{2} u + \frac{B}{\sigma + 1} u^\sigma
\]

\[
\frac{P}{n_0} = \frac{u^2 d(E/A)}{du} = \frac{2}{3} \langle E_F^0 \rangle u^{5/3} + \frac{A}{2} u^2 + \frac{B\sigma}{\sigma + 1} u^{\sigma + 1}
\]

\[
\frac{K}{9} = \frac{d(P/n_0)}{du} = \frac{10}{9} \langle E_F^0 \rangle u^{2/3} + Au + B\sigma u^\sigma.
\]

Above, \( \langle E_F^0 \rangle = (3/5)(\hbar k_F^0)^2/2m \) is the mean kinetic energy per particle at equilibrium, with \( k_F^0 \) denoting the Fermi momentum at equilibrium.
Schematic Nuclear Matter EOS-III

Setting $u = n/n_0 = 1$ for which $P/n_0 = 0$, we find

$$\sigma = \frac{K_0 + 2\langle E_F^0 \rangle}{9 \left[ \frac{1}{3} \langle E_F^0 \rangle - (\frac{E}{A} - m) \right]}$$

$$B = \left( \frac{\sigma + 1}{\sigma - 1} \right) \left[ \frac{1}{3} \langle E_F^0 \rangle - \left( \frac{E}{A} - m \right) \right]$$

$$A = \left[ \left( \frac{E}{A} - m \right) - \frac{5}{3} \langle E_F^0 \rangle \right] - B$$

Choosing $E/A - m = -16$ MeV and $n_0 = 0.16$ fm$^{-3}$ leads to:

<table>
<thead>
<tr>
<th>$K_0$ (MeV)</th>
<th>$A$ (MeV)</th>
<th>$B$ (MeV)</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>-366.23</td>
<td>313.39</td>
<td>1.16</td>
</tr>
<tr>
<td>400</td>
<td>-122.17</td>
<td>65.39</td>
<td>2.11</td>
</tr>
</tbody>
</table>
Limitations to watch out for

- Since $\sigma > 1$ for both choices of $K_0$, the energy density varies more rapidly than $n^2$ which leads to an acausal behavior in the EOS.
- Also, input values of $K_0$ below a certain limit are not allowed.

An adequate parametrization that reproduces the nuclear matter energy density of realistic microscopic calculations is provided by

$$
\epsilon_{nm} = mn_0u + \frac{3}{5}E_F^0n_0u^{5/3} + V(u),
$$

where the potential contribution $V(u)$ is given by

$$
V(u) = \frac{1}{2}An_0u^2 + \frac{Bn_0u^{\sigma+1}}{1 + B'u^{\sigma-1}} + u \sum_{i=1,2} C_i 4 \int \frac{d^3k}{(2\pi)^3} g(k, \Lambda_i) \theta(k_F - k).
$$

The function $g(k, \Lambda_i)$ is suitably chosen to simulate finite range effects.
Neutron-rich Matter-I

- \( \alpha = (n_n - n_p)/n \) := excess neutron fraction
- \( n = n_n + n_p \) := total baryon density
- \( x = n_p/n = (1 - \alpha)/2 \) := proton fraction

The neutron and proton densities are then

\[
n_n = \frac{(1 + \alpha)}{2} n = (1 - x) n \quad \& \quad n_p = \frac{(1 - \alpha)}{2} n = x n.
\]

For nuclear matter, \( \alpha = 0 \) \( (x = 0.5) \), whereas, for pure neutron matter, \( \alpha = 1 \) \( (x = 0) \).

Write the energy per particle (by simplifying \( E/A \) to \( E \)) as

\[
E(n, \alpha) = E(n, \alpha = 0) + \Delta E_{\text{kin}}(n, \alpha) + \Delta E_{\text{pot}}(n, \alpha), \quad \text{or}
\]

- 1st term := energy of symmetric nuclear matter
- 2nd & 3rd terms := isospin asymmetric parts of kinetic and interaction terms in the many-body hamiltonian
Neutron-rich Matter-II

In a non–relativistic description,

\[ \epsilon_{kin}(n, \alpha) = \frac{3}{5} \frac{\hbar^2}{2m} \left[ (3\pi^2 n_n)^{2/3} n_n + (3\pi^2 n_p)^{2/3} n_p \right] \]

\[ = n \langle E_F \rangle \cdot \frac{1}{2} \left[ (1 + \alpha)^{5/3} + (1 - \alpha)^{5/3} \right] . \]

- \( \langle E_F \rangle = (3/5)(\hbar^2/2m)(3\pi^2 n/2)^{2/3} \) := mean K.E. of nuclear matter.

\[ \Delta E_{kin}(n, \alpha) = E_{kin}(n, \alpha) - E_{kin}(n, \alpha = 0) \]

\[ = \frac{1}{3} E_F \cdot \alpha^2 \left( 1 + \frac{\alpha^2}{27} + \cdots \right) . \]

- Quadratic term above offers a useful approximation ;
- From experiments, bulk symmetry energy \( \simeq 30 \text{ MeV} \) ;
- Contribution from K.E. amounts to \( E_F^0/3 \simeq (12 - 13) \text{ MeV} \) ;
- Interactions contribute more to the total bulk symmetry energy .
Neutron-rich Matter-III

\[ E(n, x) = E(n, 1/2) + S_2(n) (1 - 2x)^2 + S_4(n) (1 - 2x)^4 + \cdots. \]

- \( S_2(n), S_4(n), \cdots \) from microscopic calculations.

Chemical Potentials:

Utilizing \( E = \epsilon/n, n = n_n + n_p, x = n_p/n, \) and \( u = n/n_0, \)

\[
\mu_n = \left. \frac{\partial \epsilon}{\partial n_n} \right|_{n_p} = E + u \left. \frac{\partial E}{\partial u} \right|_x - x \left. \frac{\partial E}{\partial x} \right|_n,
\]

\[
\mu_p = \left. \frac{\partial \epsilon}{\partial n_p} \right|_{n_n} = \mu_n + \left. \frac{\partial E}{\partial x} \right|_n,
\]

\[
\hat{\mu} = \mu_n - \mu_p = \left. -\frac{\partial E}{\partial x} \right|_n
\]

\[
= 4(1 - 2x) \left[ S_2(n) + 2S_4(n) (1 - 2x)^2 + \cdots \right].
\]

- \( \hat{\mu} \) determines the composition of charge neutral neutron star matter.
- \( \hat{\mu} \) governed by the density dependence of the symmetry energy.
Charge neutral neutron-rich matter-I

- Old neutron stars are in equilibrium w.r.t. weak interactions.

- The ground state consists of strongly interacting hadrons and weakly interacting leptons generated in $\beta$-decays and inverse $\beta$-decays.

- Local charge neutrality and chemical equilibrium prevail.

First consider matter with neutrons, protons and electrons involved in

$$n \ (or \ + \ n) \ \rightarrow \ p \ (or \ + \ n) + e^- + \bar{\nu}_e ,$$

$$p \ (or \ + \ n) + e^- \ \rightarrow \ n \ (or \ + \ n) + \nu_e$$

- In cold catalyzed neutron star matter, neutrinos leave the system leading to

$$\hat{\mu} = \mu_n - \mu_p = \mu_e .$$

(Later we will also consider the case in which neutrinos are trapped in matter, which leads instead to $\mu_n - \mu_p = \mu_e - \mu_{\nu_e}$.)
In beta equilibrium, one has
\[ \frac{\partial}{\partial x} \left[ E_b(n, x) + E_e(x) \right] = 0. \]

Charge neutrality implies that \( n_e = n_p = nx \), or, \( k_{F_e} = k_{F_p} \).

Combining these results, \( \tilde{x}(n) \) is determined from
\[ 4(1 - 2x) \left[ S_2(n) + 2S_4(n) (1 - 2x)^2 + \cdots \right] = \hbar c \left( 3\pi^2 nx \right)^{1/3}. \]

When \( S_4(n) << S_2(n) \), \( \tilde{x} \) is obtained from \( \beta \tilde{x} - (1 - 2\tilde{x})^3 = 0 \), where \( \beta = 3\pi^2 n \left( \hbar c / 4S_2 \right)^3 \). Analytic solution ugly!

For \( u \leq 1 \), \( \tilde{x} << 1 \), and to a good approximation \( \tilde{x} \simeq (\beta + 6)^{-1} \).

Notice the high sensitivity to \( S_2(n) \), which favors the addition of protons to matter.
### Charge neutral neutron-rich matter-III

**Muons in matter:**
When \( E_{Fe} \geq m_\mu c^2 \sim 105 \text{ MeV} \), electrons to convert to muons through

\[
e^- \rightarrow \mu^- + \bar{\nu}_\mu + \nu_e.
\]

Chemical equilibrium implies \( \mu_\mu = \mu_e \).
At threshold, \( \mu_\mu = m_\mu c^2 \sim 105 \text{ MeV} \).

As the proton fraction at nuclear density is small, \( 4S_2(u)/m_\mu c^2 \sim 1 \).
Using \( S_2(u = 1) \simeq 30 \text{ MeV} \), threshold density is \( \sim n_0 = 0.16 \text{ fm}^{-3} \).

Above threshold,

\[
\mu_\mu = \sqrt{k_{F\mu}^2 + m_\mu^2 c^4} = \sqrt{(\hbar c)^2 (3\pi^2 nx_\mu)^{2/3} + m_\mu^2 c^4}.
\]

- \( x_\mu = n_\mu/n_b := \text{muon fraction in matter} \).
The new charge neutrality condition is \( n_e + n_\mu = n_p \).
Muons make \( x_e = n_e/n_b \) to be lower than its value without muons.
**Charge neutral neutron-rich matter-IV**

**Total energy density & pressure:**

\[
\epsilon_{\text{tot}} = \epsilon_b + \sum_{\ell=e^-,\mu^-} \epsilon_{\ell} \quad \& \quad P_{\text{tot}} = P_b + \sum_{\ell=e^-,\mu^-} P_{\ell}
\]

- \( \epsilon_{b,\ell} \) and \( P_{b,\ell} \) := energy density and pressure of baryons (leptons).

\[
\epsilon_{\ell} = 2 \int \frac{d^3 k}{(2\pi)^3} \sqrt{k^2 + m_{\ell}^2} \quad \& \quad P_{\ell} = \frac{1}{3} \cdot 2 \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{\sqrt{k^2 + m_{\ell}^2}}
\]

\[
\epsilon_b = mn_0 u + \left\{ \frac{3}{5} E_F^0 n_0 u^{5/3} + V(u) \right\} + n_0 (1 - 2x)^2 u S(u),
\]

\[
P_b = \left\{ \frac{2}{5} E_F^0 n_0 u^{5/3} + \left( u \frac{dV}{du} - V \right) \right\} + n_0 (1 - 2x)^2 u^2 \frac{dS}{du}.
\]

- As \( \alpha_{em} \simeq 1/137 \), free gas expressions for leptons are satisfactory.
### Charge neutral neutron-rich matter-V

**STATE VARIABLES AT NUCLEAR DENSITY**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Nuclear matter</th>
<th>Stellar matter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{x}$</td>
<td>0.5</td>
<td>0.037</td>
</tr>
<tr>
<td>$\epsilon_b/n - m$</td>
<td>−16</td>
<td>9.6</td>
</tr>
<tr>
<td>$\epsilon_e/n$</td>
<td>0</td>
<td>3.18</td>
</tr>
<tr>
<td>$P_b$</td>
<td>0</td>
<td>3.5</td>
</tr>
<tr>
<td>$P_e$</td>
<td>0</td>
<td>0.17</td>
</tr>
<tr>
<td>$\mu_n - m$</td>
<td>−16</td>
<td>35.74</td>
</tr>
<tr>
<td>$\mu_p - m$</td>
<td>−16</td>
<td>−75.14</td>
</tr>
<tr>
<td>$\mu_e = \mu_n - \mu_p$</td>
<td>0</td>
<td>110.88</td>
</tr>
</tbody>
</table>

Energies in MeV and pressure in MeV fm$^{-3}$. The numerical estimates are based on an assumed symmetry energy $S_2(u) = 13u^{2/3} + 17u$, where $u = n/n_b$. 
Nucleonic Equation of State

- Energy \( (E) \) & Pressure \( (P) \) vs. scaled density \( (u = n/n_0) \).
- Nuclear matter equilibrium density \( n_0 = 0.16 \text{ fm}^{-3} \).
- Proton fraction \( x = n_p/(n_p + n_n) \).
- Nuclear matter : \( x = 1/2 \).
- Neutron matter : \( x = 0 \).
- Stellar matter in \( \beta- \) equilibrium : \( x = \tilde{x} \).
Mass Radius Relationship
